Leontief Input–Output Model: The Game Theory Approach*

Rough Draft would be an overstatement

Ben Galin

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1 Foundations: Leontief Input–Output Model

Leontief Input–Output Model is a model that analyzes the production and consumption in the economy as a whole. We are not interested in viewing the economy from the industries' perspective and even less so, from the individual firms' perspective. The model is used to analyze national, and even global, economies. The need for this model stems from the need to know how much goods and services we should produce in order to satisfy consumers' demand. The irony is apparent: while the model relies on a market economy to use prices as signals, by the very use of it we imply a central planning of the economy.

1.1 Exchange Tables

The process of generating Leontief Input–Output Model starts with a statistical survey of the economy. The economy is divided into sectors, or industries[1], and we survey two quantities in particular: what the total output of each sector is; and what and how much inputs each sector uses.

A problem arises when we realize that outputs of one industry serve as inputs of other (and quite commonly—the same) industries. As industries are depended upon other industries in order to produce their own output, they demand a greater production of services and goods. We call the collection of those demands the *intermediate demand*.

intermediate demand

The data collected by this survey is presented in an *input–output table* or an *exchange table*. In addition to the output and inputs of each industry, we also look at the consumption of the *open sector*, the only sector that consumes goods and services without producing any.

input–output table

exchange table

open sector

[1]We will use these two terms interchangeably.

Let us consider a simplified economy with just three industries: coal, electricity, and steel. Table 1 presents a possible[2] exchange table for that economy.

	Coal	Electricity	Steel	Open Sector
Coal	0	140	30	30
Electricity	50	60	60	330
Steel	80	40	30	150
Total Gross Output	200	500	300	

Table 1: Annual Exchange of Goods and Services in a Simplified Economy (in billions of US$)

The table tells us, for instance, that the coal industry consumes $80 billion worth of steel, or that the steel industry uses $30 billion worth of its own output. Note that the total gross output at the bottom of each sector's *column* is simply the sum of the output along the sector's *row*.

1.2 Consumption Matrices

consumption matrix

The next step is to build a *consumption matrix* from the data presented in the exchange table. We start by dividing each sector's input by its total gross output. Table 2 shows the outcome of that step.

	Coal	Electricity	Steel
Coal	0.00	0.28	0.10
Electricity	0.25	0.12	0.20
Steel	0.40	0.08	0.10

Table 2: Inputs Consumed Per Unit of Sector Output

productivity

This representation allows us to measure the *productivity* of each industry; if the entries in a column of a particular industry sum up to less than one, the industry is productive. Otherwise, it is not. In the scope of our discussion, we will limit ourselves to productive industries alone. Such restriction is feasible because it assumes that industries that are not productive[3] will have to either reduce their inputs or raise the prices of their outputs in order to survive. In dealing with national markets, the market also has the option to turn to the international market for alternatives.

Reading across the rows, however, takes a different meaning. If before we could sum up each row and figure how much of each industry's output goes

[2] And by *possible* we mean, of course, not possible at all; the figures do, however, sum up nicely while illustrating our arguments.

[3] Note that our definition of *productivity* differs significantly from other commonly used definitions.

for intermediate demand, such summation is no longer possible. [[[More on the meaning of rows. –Ben]]]

Considering the two previous observations, the nature of this table gives rise to the use of column vectors in a matrix as a mean of representing our findings. The consumption matrix C is as follows:

$$C = \begin{bmatrix} 0.00 & 0.28 & 0.10 \\ 0.25 & 0.12 & 0.20 \\ 0.40 & 0.08 & 0.10 \end{bmatrix} .$$

We let the *final demand* $\mathbf{d}$ be a vector representing the open sector entries in Table 1:

final demand

$$\mathbf{d} = \begin{bmatrix} 30 \\ 330 \\ 150 \end{bmatrix} .$$

If we denote the *production vector* as a column vector $\mathbf{x}$, we can conclude that the intermediate demand is nothing but $C\mathbf{x}$: the consumption matrix multiplied on the right by the production vector.

production vector

The key here is efficiency. We are interested in the production vector $\mathbf{x}$ that will satisfy the intermediate and final demands but will not involve wasted resources. Therefore, we can conclude that $\mathbf{x}$ must satisfy

$$\mathbf{x} = C\mathbf{x} + \mathbf{d} .$$

Solving for $\mathbf{x}$ yields

$$\mathbf{x} = (I - C)^{-1}\mathbf{d} ,$$

where I is the identity matrix. In our example, we find that

$$\mathbf{x} = \begin{bmatrix} 200 \\ 500 \\ 300 \end{bmatrix} ,$$

which are also the entries of the total gross output row. This shouldn't come as a surprise; we started by stating what the economy is producing. The real power of this equation lies in our ability to determine how production should change to satisfy a given change in the final demand.

2 Mapping to a Mixed-Strategy Game in Game Theory

The aim of this work is to map Leontief Input–Output Model to a mixed-strategy game in Game Theory. In doing so, we need to revert back to the consumption matrix C that was presented before:

$$C = \begin{bmatrix} 0.00 & 0.28 & 0.10 \\ 0.25 & 0.12 & 0.20 \\ 0.40 & 0.08 & 0.10 \end{bmatrix} .$$

Since we assumed that all sectors are productive, we can conclude that each column sums up to less than one. However, using a *dummy variable*, we can force the sum to one. We will return to that idea shortly, but for now, let's examine the properties of this matrix.

dummy variable

If we look at each industry as a pure strategy, each column of that matrix can be viewed as a *strategy profile* of the row player. The row player randomizes among her pure strategies in the probability shown by the appropriate entry. For example, given the column player's left-most strategy (coal), the row player would elect her bottom-most strategy (steel) with a probability of 0.40, or 40%.

strategy profile

The "real-life" sense of that statement is as follows: "A production of one dollar-worth of coal requires 0.40 dollar-worth of steel." It is important to realize the difference between "one dollar-worth of" and "one unit of." We are not implying any connection between the output-units of different industries—only the money value of these outputs. For example, in order to produce one kWh of electricity, the factory would need a specific amount of coal, no matter what the price of coal might be. However, the production of one dollar-worth of electricity is greatly depended on the price of coal.

As we noted before, there is no strict relation between the entries across each row; each column spells out the needed input to produce one dollar-worth of output, but makes no claim regarding the outputs of other industries. To adjust for that fact, we will construct a *weighted consumption matrix*.

weighted consumption matrix

2.1 Weighted Consumption Matrices

Recall that the total gross output, or the production vector is

$$\mathbf{x} = \begin{bmatrix} 200 \\ 500 \\ 300 \end{bmatrix}.$$

We will weigh the production vector such that its entries sum up to one:

$$\mathbf{x}_w = \begin{bmatrix} 0.2 \\ 0.5 \\ 0.3 \end{bmatrix}.$$

We define the weighted consumption matrix as $C_w = C\mathbf{x}_w$. The computation yields

$$C_w = \begin{bmatrix} 0.00 & 0.14 & 0.03 \\ 0.05 & 0.06 & 0.06 \\ 0.08 & 0.04 & 0.03 \end{bmatrix}.$$

This computation essentially undoes the process that yielded the original consumption matrix. Note that the resultant weighted consumption matrix is nothing more than the initial exchange table, expressed in terms of relative production rather than money value.

However, the weighted consumption matrix carries a couple of important features. Firstly, we can easily read the relative amount of resources that are used to create more output, compared to the amount that is consumed by the public. If we sum up all the entries we can conclude that the economy is producing 49% of its output toward intermediate demand. The rest is produced to meet final demand. But the real beauty lies in the mapping of this model to a mixed-strategy game.

We look at the production process as if it is split to two stages: intermediate production and final production. We let the column player be the *Final Producer* FP, and the row player, the *Intermediate Producer* IP. The economy is then viewed as a repeated game. In each iteration of the game, each of the players elects a strategy (industry) according to the weighted consumption matrix. Earlier, we observed that, given the column player's left-most strategy (coal), the row player would elect her bottom-most strategy (steel) with a probability of 0.40, or 40%. Now we can conclude that the probability of the the Final Producer electing coal while the Intermediate Producer elects steel is 8%. The meaning of that assertion is simply noting that the economy drives 8% of its total production to "create" coal out of steel. We will use the weighted consumption matrix to construct a strategy profile for each of the players.

Final Producer

Intermediate Producer

The Intermediate Producer's strategy profile is a column vector in which each entry is a summation of the respective row in the weighted consumption matrix. We will add a fourth entry that corresponds to our dummy variable: the consumer demand. The resultant vector $\mathbf{s}_{\mathrm{IP}}$ will be

$$\mathbf{s}_{\mathrm{IP}} = \begin{bmatrix} 0.17 \\ 0.17 \\ 0.15 \\ 0.51 \end{bmatrix}.$$

Similarly, we can construct the Final Producer's strategy profile $\mathbf{s}_{\mathrm{FP}}$:

$$\mathbf{s}_{\mathrm{FP}} = \begin{bmatrix} 0.13 \\ 0.24 \\ 0.12 \\ 0.51 \end{bmatrix}.$$

2.2 Payoff Matrices

So we have defined strategy profiles for both players, but in order to make any use of this model, we need to identify a *payoff matrix*. Recall that a two-player payoff matrix is a table of ordered pairs in which the first element is the *payoff*, or *utility* for the row player and second element is the payoff for the column player. The assumption is that each player will pick a strategy that maximizes his payoff. We aim to simplify this model as much as possible, so our intention is to build a payoff matrix that describes a *zero-sum game*, a game in which one player's gain is the other's loss. The intuitive sense of using a zero-sum game is as follows: the Final Producer will try to minimize production. The Intermediate Producer, on the other hand, will try to maximize production. Right in between lies our equilibrium point.

payoff matrix

payoff

utility

zero-sum game

Once again, for simplicity's sake, we would prefer to use *Nash Equilibria* as our equilibrium points. Loosely speaking, a Nash Equilibrium is achieved if no player can benefit by changing his strategy while the other players keep their strategies unchanged. An important concept to comprehend here is that no player would mix two (or more) strategies unless she is indifferent between them. Specifically, given FP's strategy profile, IP's *expected utility* from each of her strategies should be the same. For example, consider the famous *Battle of the Sexes* game presented in Table 3.

Nash Equilibrium

expected utility

	X	Y
A	(2,1)	(0,0)
B	(0,0)	(1,2)

Table 3: Battle of the Sexes Game

The two Nash Equilibria of this game are $\{\mathrm{A}, \mathrm{X}\}$ and $\{\mathrm{B}, \mathrm{Y}\}$ as no player can do better by changing his strategy. However, we are interested in the third Nash Equilibrium of this game: a mixed-strategy equilibrium.

Let $U_r(\mathrm{A})$ be the row player's expected utility from choosing strategy A. We assume that the column player randomizes between his two strategies X and Y with the probabilities p and $1-p$, respectively. $U_r(\mathrm{A})$, then, is calculated by summing the payoffs for the row player across strategy A, weighted by the respective probabilities of each payoff. Namely,

$$U_r(\mathrm{A}) = 2 \cdot p + 0 \cdot (1-p) \,.$$

Similarly,

$$U_r(\mathrm{B}) = 0 \cdot p + 1 \cdot (1-p) \,.$$

Since we know that $U_r(\mathrm{A}) = U_r(\mathrm{B})$ (or otherwise she wouldn't have mixed the two pure strategies), we can solve both equations simultaneously. The computation yields $p = 1/3$ and $U_r(\mathrm{A}) = U_r(\mathrm{B}) = 2/3$. By symmetry, one can easily find the expected utility for the column player. The strategy profile for the row player is

$$\mathbf{s}_r = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} ,$$

and for the column player,

$$\mathbf{s}_c = \begin{bmatrix} 1/3 \\ 2/3 \end{bmatrix} .$$

Let's return to our model. We will use the fact that we know the probabilities in which each player elects his strategies to reconstruct the payoff matrix. For clarity, we will start with a parameterized payoff matrix as illustrated in Table 4. Given IP's strategy i and FP's strategy j, each entry $P_{i,j}$ is the payoff for IP. Note that since this is a zero-sum game, the payoff for FP is $-Pi, j$.

	Coal	Electricity	Steel	Dummy
Coal	$P_{1,1}$	$P_{1,2}$	$P_{1,3}$	$P_{1,4}$
Electricity	$P_{2,1}$	$P_{2,2}$	$P_{2,3}$	$P_{2,4}$
Steel	$P_{3,1}$	$P_{3,2}$	$P_{3,3}$	$P_{3,4}$
Dummy	$P_{4,1}$	$P_{4,2}$	$P_{4,3}$	$P_{4,4}$

Table 4: Parameterized Payoff Matrix

Recall that the strategy profiles for IP and FP are

$$\mathbf{s}_{\mathrm{IP}} = \begin{bmatrix} 0.17 \\ 0.17 \\ 0.15 \\ 0.51 \end{bmatrix} \quad \text{and} \quad \mathbf{s}_{\mathrm{FP}} = \begin{bmatrix} 0.13 \\ 0.24 \\ 0.12 \\ 0.51 \end{bmatrix},$$

respectively. The expected utility for IP is

$$\begin{aligned} U_{\mathrm{IP}} &= 0.13 \cdot P_{1,1} \ + \ 0.24 \cdot P_{1,2} \ + \ 0.12 \cdot P_{1,3} \ + \ 0.51 \cdot P_{1,4} \\ &= 0.13 \cdot P_{2,1} \ + \ 0.24 \cdot P_{2,2} \ + \ 0.12 \cdot P_{2,3} \ + \ 0.51 \cdot P_{2,4} \\ &= 0.13 \cdot P_{3,1} \ + \ 0.24 \cdot P_{3,2} \ + \ 0.12 \cdot P_{3,3} \ + \ 0.51 \cdot P_{3,4} \\ &= 0.13 \cdot P_{4,1} \ + \ 0.24 \cdot P_{4,2} \ + \ 0.12 \cdot P_{4,3} \ + \ 0.51 \cdot P_{4,4} \,. \end{aligned}$$

And for FP, the expected utility is

$$\begin{aligned} U_{\mathrm{FP}} &= -0.17 \cdot P_{1,1} \ - \ 0.17 \cdot P_{2,1} \ - \ 0.15 \cdot P_{3,1} \ - \ 0.51 \cdot P_{4,1} \\ &= -0.17 \cdot P_{1,2} \ - \ 0.17 \cdot P_{2,2} \ - \ 0.15 \cdot P_{3,2} \ - \ 0.51 \cdot P_{4,2} \\ &= -0.17 \cdot P_{1,3} \ - \ 0.17 \cdot P_{2,3} \ - \ 0.15 \cdot P_{3,3} \ - \ 0.51 \cdot P_{4,3} \\ &= -0.17 \cdot P_{1,4} \ - \ 0.17 \cdot P_{2,4} \ - \ 0.15 \cdot P_{3,4} \ - \ 0.51 \cdot P_{4,4} \,. \end{aligned}$$

We have eight equations with eighteen unknowns, so we are free to set ten of them to zero. For reasons that will be discussed shortly, we will set $P_{1,1}$ as one and $P_{1,3}$, $P_{1,4}$, $P_{2,2}$, $P_{2,4}$, $P_{3,1}$, $P_{3,3}$, $P_{4,1}$, $P_{4,2}$, and $P_{4,4}$ as zero. The new system of equations is as follows:

$$\begin{cases} U_{\mathrm{IP}} &= & 0.24 \cdot P_{1,2} + 0.13 \\ U_{\mathrm{IP}} &= & 0.13 \cdot P_{2,1} + 0.12 \cdot P_{2,3} \\ U_{\mathrm{IP}} &= & 0.24 \cdot P_{3,2} + 0.51 \cdot P_{3,4} \\ U_{\mathrm{IP}} &= & 0.12 \cdot P_{4,3} \\ U_{\mathrm{FP}} &= & -0.17 \cdot P_{2,1} - 0.17 \\ U_{\mathrm{FP}} &= & -0.17 \cdot P_{1,2} - 0.15 \cdot P_{3,2} \\ U_{\mathrm{FP}} &= & -0.17 \cdot P_{2,3} - 0.51 \cdot P_{4,3} \\ U_{\mathrm{FP}} &= & -0.15 \cdot P_{3,4} \end{cases} .$$

Solving simultaneously yields

$$\begin{cases} P_{1,2} & \approx & -0.7558 \\ P_{2,1} & \approx & -1.3023 \\ P_{2,3} & \approx & 0.9826 \\ P_{3,2} & \approx & 0.5140 \\ P_{3,4} & \approx & -0.3426 \\ P_{4,3} & \approx & -0.4283 \\ U_{\mathrm{IP}} & \approx & -0.0514 \\ U_{\mathrm{FP}} & \approx & 0.0514 \end{cases} .$$

Our new payoff matrix is presented in Table 5. [[Draft only. I am thinking of parameterizing it differently. -Ben]]

	Coal	Electricity	Steel	Dummy
Coal	1	-0.7558	0	0
Electricity	-1.3023	0	0.9826	0
Steel	0	0.5140	0	-0.3426
Dummy	0	0	-0.4283	0

Table 5: Payoff Matrix