Classification of the Irreducible Representations of the Dihedral Group $D_{2n}$

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Let $D_{2n}$ be the dihedral group with $2n$ elements, where $n \geq 3$, corresponding to rigid transformation of a regular $n$-gon. In this paper, we classify the irreducible representations of $D_{2n}$ and their corresponding irreducible $D_{2n}$-modules.

Typically, one writes the presentation $D_{2n} = \langle a, b \mid a^n = b^2 = 1, b^{-1}ab = a^{-1} \rangle$, where the generator $a$ corresponds to a clockwise rotation of $\frac{2}{n}$ and the generator $b$ corresponds to a reflection over a fixed axis through one of the vertices. It turns out that for our purpose it is more convenient to think of $D_{2n}$ with the presentation $\langle s_1, s_2 \mid s_2 s_1 = s_2^2 = (s_1 s_2)^n = 1 \rangle$.

Theorem. Let $D_{2n}$ be the dihedral group with $2n$ elements. We distinguish two cases:

1. $n = 2k + 1$ is odd. Up to equivalency of representations, the irreducible representations of $D_{2n}$ are

$$
\begin{align*}
\rho_0 : D_{2n} &\to GL_1(\mathbb{C}) \\
s_1 &\mapsto (1) \\
s_2 &\mapsto (1)
\end{align*}
$$

$$
\begin{align*}
\rho_{-1} : D_{2n} &\to GL_1(\mathbb{C}) \\
s_1 &\mapsto (-1) \\
s_2 &\mapsto (-1)
\end{align*}
$$

$$
\begin{align*}
\rho_i : D_{2n} &\to GL_2(\mathbb{C}) \\
s_1 &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
s_2 &\mapsto \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_i & 0 \end{pmatrix},
\end{align*}
$$

where $\omega_n = e^{\frac{2\pi i}{n}}$ is the principal $n$th root of unity, and $i : 1, \ldots, k$.

The corresponding $D_{2n}$-modules are $V_{-1}, \ldots, V_k$, such that if $\{v^{(0)}\}$, $\{v^{(-1)}\}$, and $\{v_1^{(i)}, v_2^{(i)}\}$ are the bases for $V_0$, $V_{-1}$, and $V_i$, respectively, and where $i : 1, \ldots, k$, then the following actions\(^1\) define the $D_{2n}$-modules:

$$
\begin{align*}
\begin{cases}
v^{(0)} \circ s_1 = v^{(0)} \\
v^{(0)} \circ s_2 = v^{(0)}
\end{cases}
\quad
\begin{cases}
v^{(-1)} \circ s_1 = -v^{(-1)} \\
v^{(-1)} \circ s_2 = -v^{(-1)}
\end{cases}
\quad
\begin{cases}
v_1^{(i)} \circ s_1 = v_2^{(i)} \\
v_1^{(i)} \circ s_2 = \omega^{-i} v_2^{(i)}
\end{cases}
\quad
\begin{cases}
v_2^{(i)} \circ s_1 = v_1^{(i)} \\
v_2^{(i)} \circ s_2 = \omega^{i} v_1^{(i)}
\end{cases}
\end{align*}
$$

\(^1\)With a slight abuse of notation, we denote the actions $V_i \times D_{2n} \to V_i$ by $\circ$ for all $i : -1, \ldots, k$. 

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2. \( n = 2k \) is even. Up to equivalency of representations, the irreducible representations of \( D_{2n} \) are

\[
\begin{align*}
\rho_0 : D_{2n} &\rightarrow GL_1(\mathbb{C}) & \rho_{-1} : D_{2n} &\rightarrow GL_1(\mathbb{C}) & \rho_{-2} : D_{2n} &\rightarrow GL_1(\mathbb{C}) & \rho_{-3} : D_{2n} &\rightarrow GL_1(\mathbb{C}) \\
s_1 &\mapsto (1) & s_1 &\mapsto (1) & s_1 &\mapsto (-1) & s_1 &\mapsto (-1) \\
s_2 &\mapsto (1) & s_2 &\mapsto (-1) & s_2 &\mapsto (1) & s_2 &\mapsto (-1) \\
\rho_i : D_{2n} &\rightarrow GL_2(\mathbb{C}) \\
s_1 &\mapsto \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\
s_2 &\mapsto \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix},
\end{align*}
\]

where \( \omega_n = e^{\frac{2\pi i}{n}} \) is the principal \( n \)th root of unity, and \( i : 1, \ldots, k - 1 \).

The corresponding \( D_{2n} \)-modules are \( V_{-3}, \ldots V_{k-1} \), such that if \( \{v^{(0)}\}, \{v^{(-1)}\}, \{v^{(-2)}\}, \{v^{(-3)}\} \), and \( \{v_1^{(i)}, v_2^{(i)}\} \) are the bases for \( V_0, V_{-1}, V_2, V_{-3}, \) and \( V_i \), respectively, and where \( i : 1, \ldots, k - 1 \), then the following actions define the \( D_{2n} \)-modules:

\[
\begin{align*}
v^{(0)} \circ s_1 &= v^{(0)} & v^{(-1)} \circ s_1 &= v^{(-1)} & v^{(-2)} \circ s_1 &= -v^{(-2)} & v^{(-3)} \circ s_1 &= -v^{(-3)} \\
v^{(0)} \circ s_2 &= v^{(0)} & v^{(-1)} \circ s_2 &= -v^{(-1)} & v^{(-2)} \circ s_2 &= v^{(-2)} & v^{(-3)} \circ s_2 &= -v^{(-3)} \\
v_1^{(i)} \circ s_1 &= v_2^{(i)} & v_2^{(i)} \circ s_1 &= v_1^{(i)} \\
v_1^{(i)} \circ s_2 &= \omega_n^{-i}v_2^{(i)} & v_2^{(i)} \circ s_2 &= \omega_n^i v_1^{(i)}.
\end{align*}
\]

Before we proceed with the proof of the theorem, we outline the crucial steps of the proof. We will show that (i) the above maps are indeed representations of \( D_{2n} \); (ii) they are irreducible; (iii) no two of them are equivalent; and (iv) they exhaust the list of non-equivalent \( D_{2n} \) representations. The proofs of (ii) and (iii) will not be directly on the representations \( \rho_i \), but rather on the corresponding \( D_{2n} \)-modules. Such a method is valid since a representation of \( D_{2n} \) is irreducible if and only if its corresponding \( D_{2n} \)-module is irreducible, and two representations are equivalent if and only if their corresponding \( D_{2n} \)-modules are isomorphic. The proof of (iv) will also look at the corresponding \( D_{2n} \)-modules and will invoke a counting argument.

Since the statement of the theorem splits into two cases—when \( n \) is odd and when \( n \) is even—we will also consider the two cases separately. Fortunately, most of the arguments for one case are applicable to the other one with only a few minor changes.

**Proof.** Suppose \( n = 2k + 1 \) is odd. Our first step (i) of the proof is showing that the maps \( \rho_i \) are representations. To that end, we need to check that the \( \rho_i \)s are group homomorphisms. It suffices to check that the relations on the generators \( s_1 \) and \( s_2 \) are preserved under the maps. That is, we need to verify that \( (s_1)\rho_i(s_1)\rho_i = (s_2)\rho_i(s_2)\rho_i = I \) and \( [(s_1)\rho_i(s_2)\rho_i]^{k} (s_1)\rho_i = (s_2)\rho_i [(s_1)\rho_i(s_2)\rho_i]^{k} \), for \( i = -1, \ldots, k \) and where \( I \) is the identity matrix.

Consider first \( i = 0 \). Then \( (s_1)\rho_0(s_1)\rho_0 = (1)(1) = (1) \) and \( (s_2)\rho_0(s_2)\rho_0 = (1)(1) = (1) \).
In addition, \[ (s_1)\rho_0(s_2)\rho_0^k \rho_0 = (1) = (s_2)\rho_0 [(s_1)\rho_0(s_2)\rho_0]^k. \] Similarly, if \( i = -1 \), then \((s_1)\rho_{-1}(s_2)\rho_{-1} = (-1)(-1) = (1)\) and \((s_2)\rho_{-1}(s_2)\rho_{-1} = (-1)(-1) = (1)\). As before, we also have \([s_2]_k\rho_{-1}(s_2)\rho_{-1}^k = (1) = (s_2)\rho_{-1} [(s_1)\rho_{-1}(s_2)\rho_{-1}]^k\). Consider now \( 1 \leq i \leq k \).

Clearly, we have
\[
(s_1)\rho_i(s_1)\rho_i = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad (s_2)\rho_i(s_2)\rho_i = \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_n^{ik} \\ \omega_n^{-ik} & 0 \end{pmatrix}.
\]

For the alternating product, note first that \( i(2k+1) \equiv 0 \pmod{n}, \) so \( ik \equiv -i - ik \pmod{n} \). Hence, \( \omega_n^{ik} = \omega_n^{-i-ik} \), so we have
\[
[(s_1)\rho_i(s_2)\rho_i]^k(s_1)\rho_i = \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} \right]^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \left[ \begin{pmatrix} \omega_n^i & 0 \\ 0 & \omega_n^{-i} \end{pmatrix} \right]^k \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_n^{ik} \\ \omega_n^{-ik} & 0 \end{pmatrix} = \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} \left[ \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & \omega_n^{-i} \\ \omega_n^i & 0 \end{pmatrix} \right]^k = (s_2)\rho_i [(s_1)\rho_i(s_2)\rho_i]^k.
\]

We conclude that the maps above are indeed representations.

Next (ii) we show each map \( \rho_i \) is irreducible, where \( i : -1, \ldots, k \). Recall that by definition a representation \( \rho \) of \( D_{2n} \) is irreducible if and only if its corresponding \( D_{2n} \)-module with respect to some fixed basis is irreducible. The dimension of a nonzero submodule \( W \subset V \) must be strictly less than the dimension of \( V \) and strictly greater than zero. The \( D_{2n} \)-modules \( V_{-1} \) and \( V_0 \) are one-dimensional, so clearly they are irreducible.

It remains to check the \( V_i \) is irreducible for \( i : 1, \ldots, k \). Note that if a two-dimensional \( D_{2n} \)-module \( V_i \) is reducible, then it must have a one-dimensional \( D_{2n} \)-submodule \( W \). That is, \( W = \mathbb{C} \text{-span}\{\alpha v_1^{(i)} + \beta v_2^{(i)}\} \), for some complex numbers \( \alpha \) and \( \beta \), where at least one of \( \alpha, \beta \) is nonzero. It follows that there exists a nonzero complex number \( \gamma \) for which
\[
(\alpha v_1^{(i)} + \beta v_2^{(i)}) \circ s_1 = \beta v_1^{(i)} + \alpha v_2^{(i)} = \gamma (\alpha v_1^{(i)} + \beta v_2^{(i)}).
\]

Hence, \( \alpha = \beta = 1 \) for simplicity. Since \( W \) is a \( D_{2n} \)-submodule, there exists a nonzero complex number \( \delta \) for which
\[
(v_1^{(i)} + v_2^{(i)}) \circ s_2 = \omega_n^i v_1^{(i)} + \omega_n^{-i} v_2^{(i)} = \delta (v_1^{(i)} + v_2^{(i)}).
\]

This is possible if and only if \( \omega_n^i = \omega_n^{-i} \). But this is the case if and only if \( i \equiv -i \pmod{n} \), which is equivalent to requiring \( 2i \equiv 0 \pmod{n} \). But since \( 2 \leq 2i \leq n - 1 \), we conclude that a one-dimensional \( D_{2n} \)-module \( W \) does not exist. We have shown that the representations \( \rho_i \) are irreducible for \( i : -1, \ldots, k \).
Next (iii) we show that \( \rho_i \) and \( \rho_j \) are not equivalent for \( i \neq j \) and where \(-1 \leq i, j \leq k\). A necessary condition for two representations \( \rho : G \rightarrow GL_n(\mathbb{C}) \) and \( \sigma : G \rightarrow GL_m(\mathbb{C}) \) is \( n = m \), so we need only show that \( \rho_{-1} \) is not equivalent to \( \rho_0 \) and that \( \rho_i \) is not equivalent to \( \rho_j \) for \( i \neq j \) and \( 1 \leq i, j \leq k \).

By definition, \( \rho_{-1} \) is equivalent to \( \rho_0 \) if there exists a \( 1 \times 1 \) invertible matrix \( T \) such that for all \( s \in D_{2n} \), we have \( (s)\rho_{-1} = T^{-1}(s)\rho_0T \). In particular, for these two representations to be equivalent, there must be a nonzero complex number \( \alpha \) such that

\[
(-1) = (s_1)\rho_{-1} = (\alpha)^{-1}(s)\rho_0(\alpha) = (\alpha)^{-1}(1)(\alpha) = (\alpha)^{-1}(\alpha)(1) = (1),
\]

which clearly is impossible.

Consider now \( i \neq j \) where \( 1 \leq i, j \leq k \). Recall that \( \rho_i \) and \( \rho_j \) are equivalent if and only if their corresponding \( D_{2n} \)-modules with respect to some fixed bases are isomorphic. Consider a \( D_{2n} \)-module homomorphism \( \varphi : V_i \rightarrow V_j \) with \( v_1(i) \mapsto \alpha v_1(j) + \beta v_2(j) \) and \( v_2(i) \mapsto \gamma v_1(j) + \delta v_2(j) \) for some complex numbers \( \alpha, \beta, \gamma, \) and \( \delta \). Then

\[
v_2(i) = v_1(i) \mapsto \alpha v_1(j) + \beta v_2(j) \text{ and } v_2(i) \mapsto \gamma v_1(j) + \delta v_2(j).
\]

Hence, \( \alpha = \delta \) and \( \beta = \gamma \), so that we may write \( v_2(i) \mapsto \beta v_1(j) + \alpha v_2(j) \). Also,

\[
\omega^{-i} v_2(i) = v_1(i) \mapsto \alpha v_1(j) \text{ and } \omega^{-i} v_2(i) = v_1(j) + \alpha \omega^{-i} v_2(j).
\]

Thus, we must have \( \beta \omega_n^i = \beta \omega_n^{-i} \) and \( \alpha \omega_n^i = \alpha \omega_n^{-i} \). This is possible if and only if \( \alpha = 0 \) or \( i \equiv j \) (mod \( n \)) and \( \beta = 0 \) or \( i \equiv -j \) (mod \( n \)). Since \( 1 \leq i, j \leq k = \frac{n-1}{2} \) and \( i \neq j \), we must have \( \alpha = \beta = 0 \). It follows that \( \varphi \) is the zero \( D_{2n} \)-module homomorphism, which is not invertible. In particular, \( \varphi \) is not a \( D_{2n} \)-module isomorphism. We conclude that \( \rho_i \) is not equivalent to \( \rho_j \) for all \( i \neq j \) where \( 1 \leq i, j \leq k \).

The last (iv) remaining step is showing that the \( \rho_i \)'s we have defined above exhaust the list of non-equivalent \( D_{2n} \) representations. Here we recall that if \( V_{-1}, \ldots, V_k \) form a complete set of non-isomorphic irreducible \( D_{2n} \)-submodules, then

\[
\sum_{i=-1}^{k} (\dim V_i)^2 = |D_{2n}|.
\]

But, \( \sum_{i=-1}^{k} (\dim V_i)^2 = 2(1^2) + k(2^2) = 2(2k + 1) = 2n = |D_{2n}| \), to complete the classification in the case of \( n \) odd.

We now turn to the case \( n = 2k \) even. We start with step (i): showing that \( \rho_{-3}, \ldots, \rho_{k-1} \) are representations. We again need to check the relations on the images of the generators: \( (s_1)\rho_{i}(s_1)\rho_i = (s_2)\rho_{i}(s_2)\rho_i = I \) and \( [(s_1)\rho_{i}(s_2)\rho_i]^k = [(s_2)\rho_{i}(s_1)\rho_i]^k (s_2)\rho_i \), for \( i = -3, \ldots, k-1 \) and where \( I \) is the identity matrix.

Consider first \(-3 \leq i \leq 0\). Then \( (s_1)\rho_{i}(s_1)\rho_i = (\pm 1)(\pm 1) = (1) \) and \( (s_2)\rho_{i}(s_2)\rho_i = (\pm 1)(\pm 1) = (1) \).
In addition, 
\[ [(s_1)\rho_i(s_2)\rho_i]^k = (\pm 1)^k = [(s_2)\rho_i(s_1)\rho_i]^k. \]
Consider now \(1 \leq i \leq k - 1. \) Then 
\( (s_1)\rho_i(s_1)\rho_i = (s_2)\rho_i(s_2)\rho_i = I \) is exactly as in the case of \( n \) odd. For the alternating product, we have 
\( 2ik \equiv 0 \pmod{n}, \) so 
\( ik \equiv -ik \pmod{n}. \) Hence, 
\[ \omega_n^{ik} = \omega_n^{-ik}, \]
and we have
\[ [s_1\rho_i(s_2)\rho_i]^k = \left[ \begin{array}{cc} 1 & 0 \\ 0 & 1 \end{array} \right]^k = \left[ \begin{array}{cc} \omega_n^i & 0 \\ 0 & \omega_n^{-i} \end{array} \right]^k = \left[ \begin{array}{cc} \omega_n^{ik} & 0 \\ 0 & \omega_n^{-ik} \end{array} \right] = \left[ \begin{array}{cc} \omega_n^{-ik} & 0 \\ 0 & \omega_n^{ik} \end{array} \right]. \]

As before, we conclude that the maps above are indeed representations.

Next (ii), we need to show that the representations \( \rho_i \) are irreducible, where \( i : -3, \ldots, k - 1. \) The corresponding \( D_{2n} \)-modules \( V_{-3}, \ldots, V_0 \) are one-dimensionals and thus irreducible. Following the exact same argument as in the case \( n \) odd, the remaining \( \rho_i \) are also irreducible provided that 
\( 2i \not\equiv 0 \pmod{n}. \) But since \( 2 \leq 2i \leq 2(k - 1) = n - 2, \) this is indeed the case.

Next (iii), we show that \( \rho_i \) and \( \rho_j \) are not equivalent for \( i \neq j \) and \( -3 \leq i, j \leq k - 1. \) For \( \rho_{-3}, \ldots, \rho_0, \) using again the fact that \( 1 \times 1 \) matrices commute, we know that any two representations \( \rho_i \) and \( \rho_j \) are equivalent if 
\( (s_1)\rho_i = \rho_j \) and 
\( (s_2)\rho_i = \rho_j. \) But this does not hold for any such pair \( \rho_i, \rho_j. \) Consider now \( 1 \leq i, j \leq k - 1. \) Following the same argument as in the case \( n \) odd, two representations \( \rho_i \) and \( \rho_j \) are equivalent provided that \( i \equiv j \pmod{n} \) or \( i \equiv -j \pmod{n}. \) But again, since \( 1 \leq i, j \leq k - 1 = \frac{n - 2}{2} \) and \( i \neq j, \) we conclude that \( \rho_i \) is not equivalent to \( \rho_j \) for all \( i \neq j \) and \( 1 \leq i, j \leq k - 1. \)

Lastly (iv), we note that 
\[ \sum_{i=-3}^{k-1} = 4(1^2) + (k - 1)(2^2) = 4k = 2n = |D_{2n}|, \]
so our list of non-equivalent \( D_{2n} \) representations is complete. This concludes the proof for the case \( n \) even and proves the theorem.