

# Classification of the Irreducible Modules of the Monomial Matrix Group $G(2, 1, n)$

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Let  $G(2, 1, n) \cong C_2 \wr S_n$ , where  $n \geq 2$ , be the monomial matrix group of  $n \times n$  matrices with exactly one nonzero entry in every column and row, and the nonzero entries are in  $C_2$ , the cyclic group of order two. In this paper, we classify the irreducible representations of  $G(2, 1, n)$ .

Recall that  $G(2, 1, n)$  is generated by  $\{e_1, s_1, \dots, s_{n-1}\}$  with the following relations:

$$\begin{aligned} e_1 s_1 e_1 s_1 &= s_1 e_1 s_1 e_1, & s_i s_j &= s_j s_i \text{ for } |i - j| \geq 1, & s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, & (1) \\ e_1^2 &= s_1^2 = \dots = s_{n-1}^2 = 1. \end{aligned}$$

It is often more convenient, however, to define the elements  $e_j = s_{j-1} e_{j-1} s_{j-1}$  for  $j = 2, \dots, n$  and to think of  $G(2, 1, n)$  as generated by  $\{e_1, \dots, e_n, s_1, \dots, s_{n-1}\}$  with the relations

$$\begin{aligned} e_i e_j &= e_j e_i, & e_i s_j &= s_j e_{i s_j}, & s_i s_j &= s_j s_i \text{ for } |i - j| \geq 1, & s_i s_{i+1} s_i &= s_{i+1} s_i s_{i+1}, \\ e_1^2 &= s_1^2 = \dots = s_{n-1}^2 = 1. \end{aligned}$$

We shall switch back and forth between the two presentations of  $G(2, 1, n)$  in the following discussion.

Before we can state the classification theorem, we must extend the definitions of a *tableau of shape*  $\lambda$  and a *standard tableau of shape*  $\lambda$  to an ordered pair  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  of partitions with  $n$  total boxes. We also provide a natural extension to the definition of the *tableau content*  $C(T(i))$  of the box containing  $i$  in a standard tableau  $T$  of shape  $\lambda$  to the case when  $\lambda$  is an ordered pair  $(\lambda^{(1)}, \lambda^{(2)})$  of partitions with  $n$  total boxes.

**Definition 1.** A *tableau of shape*  $\lambda$ , where  $\lambda$  is an ordered pair  $(\lambda^{(1)}, \lambda^{(2)})$  of partitions with  $n$  total boxes, is a filling of the boxes of  $\lambda$  by  $1, 2, \dots, |\lambda^{(1)}| + |\lambda^{(2)}|$ .

**Definition 2.** Let  $T$  be a tableau of shape  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ . Then  $T$  is said to be a *standard tableau of shape*  $\lambda$  if the entries in the fillings of  $\lambda^{(1)}$  and  $\lambda^{(2)}$  each increase in rows and columns.

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**Definition 3.** Let  $T$  be a standard tableau of shape  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ . The *tableau content*  $C(T(i))$  of the box containing  $i$  in  $T$  is the content  $c_{\lambda^{(1)}}(\boxed{i})$  if the box containing  $i$  is in the filling of  $\lambda^{(1)}$  and  $c_{\lambda^{(2)}}(\boxed{i})$  otherwise.

We can now state the main theorem of this paper.

**Theorem 4.** *Let  $G(2, 1, n)$  be the monomial matrix group of  $n \times n$  matrices with exactly one nonzero entry in every column and row, and the nonzero entries are in  $C_2$ , the cyclic group of order two.*

1. *The irreducible  $G(2, 1, n)$ -modules  $V^\lambda$  are indexed by ordered pairs  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  of partitions with  $n$  total boxes.*
2. *Their dimensions are  $\dim(V^\lambda) = n_\lambda$ , where  $n_\lambda$  is the number of standard tableaux of shape  $\lambda$ .*
3. *If  $V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$ , then*

$$\begin{aligned} v_T e_i &= (-1)^{m+1} v_T, \\ v_T s_i &= C_T(i) v_T + (1 + C_T(i)) v_{T s_i}, \end{aligned}$$

where (i) the box containing  $i$  is in the partition  $\lambda^{(m)}$ ; (ii)  $v_{T s_i} = 0$  if  $v_{T s_i}$  is not a standard tableau; and (iii)  $C_T(i) = \frac{1}{C(T(i+1)) - C(T(i))}$  if the boxes containing  $i$  and  $i + 1$  are in the same partition  $\lambda^{(k)}$  and zero otherwise.

Before we proceed with the proof of the theorem, we outline the crucial steps of the proof. We will show that (i) the  $V^\lambda$  are indeed  $G(2, 1, n)$ -modules; (ii) any two different such modules are non-isomorphic; (iii) these modules are irreducible; and (iv) this classification exhausts the list of irreducible  $G(2, 1, n)$ -modules, up to isomorphism.

To prove (i), we will check that the relations of  $G(2, 1, n)$  are preserved under the action of the group on  $V^\lambda$ . The proof of (ii) will rely on generating elements analogous to the Murphy elements. We will show that their actions on a standard tableau completely determine the filling of the tableau. An immediate result of this would be (ii). For (iii), we will show that  $V^\lambda$  does not contain nonzero submodules besides itself by deriving some properties of the standard tableaux. Lastly, the proof of (iv) will involve a counting argument.

The following general observation will assist us through out the rest of the proof: The definition of a standard tableau implies that if neither of the boxes immediately to the right and immediately below a box containing  $i$  contain  $i + 1$ , then they contain values greater than  $i + 1$ . Moreover, the box containing  $i + 1$  is either in a different partitions  $\lambda^{(k)}$  or it is strictly below and strictly to the right of the box containing  $i$ . In either case, the boxes above and to the left of the box containing  $i + 1$  contain values less than  $i$ . In such a case, the tableau resulting from permuting the boxes containing  $i$  and  $i + 1$  is a standard tableau.

We summarize this result with the following remark:

*Remark 5.* Let  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  be an order pair of partitions with  $n$  total boxes, and let  $T$  be a standard partition of shape  $\lambda$ . Then whenever the boxes containing  $i$  and  $i + 1$  in  $T$  are not adjacent,  $Ts_i$  is also a standard tableau of shape  $\lambda$ . In particular, if the boxes containing  $i$  and  $i + 1$  are in different partitions  $\lambda^{(1)}$  and  $\lambda^{(2)}$ , then  $Ts_i$  is a standard tableau of shape  $\lambda$ .

Proving Theorem 4 will be significantly easier to follow if we establish a few lemmas to help us later. The first lemma proves item (i) of the outline we presented above: the  $V^\lambda$  are  $G(2, 1, n)$ -modules. The proof is rather long and tedious, though not particularly involved. In the interest of a smooth reading flow, the reader may wish to assume the lemma for the time being and return to its proof later.

**Lemma 6.** *Let  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  be an order pair of partitions with  $n$  total boxes. Then  $V^\lambda = \mathbb{C}\text{-span}\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$  is a  $G(2, 1, n)$ -module with respect to the actions defined in Theorem 4(3).*

*Proof.* We need to check that the relations defined in (1) are satisfied by the specified actions. In the rest of the proof, let  $v_T \in V^\lambda$  for a standard tableau  $T$ .

**1. The relation  $e_1^2 = 1$**

We start by showing that  $v_T e_1^2 = v_T$ . Suppose first that the box containing 1 is in  $\lambda^{(1)}$ . Then  $v_T e_1^2 = v_T e_1 = v_T$ . Otherwise, the box containing 1 is in  $\lambda^{(2)}$ . Then it follows that  $v_T e_1^2 = -v_T e_i = v_T$ , as needed.

**2. The relation  $s_i^2 = 1$**

Next, we show that  $v_T s_i^2 = v_T$  for  $1 \leq i \leq n - 1$ . We consider the following cases regarding the relative positions of the boxes containing  $i$  and  $i + 1$  in  $T$ :

- (a) the two boxes are adjacent;
- (b) the two boxes are not adjacent.

In case (a), we have  $C_T(i) = \pm 1$  and  $Ts_i$  is not standard. Thus,  $v_T s_i^2 = \pm v_T s_i = v_T$ . In case (b), we recall from Remark 5 that  $Ts_i$  is standard. We also clearly have  $C_{Ts_i}(i) = -C_T(i)$ . Hence,

$$\begin{aligned} v_T s_i^2 &= C_T(i)v_T s_i + [1 + C_T(i)]v_{Ts_i} s_i \\ &= C_T(i)^2 v_T + C_T(i)[1 + C_T(i)]v_{Ts_i} - C_T(i)[1 + C_T(i)]v_{Ts_i} + [1 - C_T(i)][1 + C_T(i)]v_T = v_T. \end{aligned}$$

### 3. The relation $s_i s_j = s_j s_i$

Next, we show that  $v_T s_i s_j = v_T s_j s_i$  for  $1 \leq i, j \leq n-1$  and  $|i-j| > 1$ . We observe that  $C_T(i) = C_{T s_j}(i)$  and  $C_T(j) = C_{T s_i}(j)$ . Thus,

$$\begin{aligned}
v_T s_i s_j &= C_T(i) v_T s_j + [1 + C_T(i)] v_{T s_i s_j} \\
&= C_T(i) C_T(j) v_T + C_T(i) [1 + C_T(j)] v_{T s_j} + [1 + C_T(i)] C_T(j) v_{T s_i} + [1 + C_T(i)] [1 + C_T(j)] v_{T s_i s_j} \\
&= C_T(j) C_T(i) v_T + C_T(j) [1 + C_T(i)] v_{T s_i} + [1 + C_T(j)] C_T(i) v_{T s_j} + [1 + C_T(j)] [1 + C_T(i)] v_{T s_j s_i} \\
&= C_T(j) v_T s_i + [1 + C_T(j)] v_{T s_j s_i} = v_T s_j s_i,
\end{aligned}$$

where we recall that for any  $\sigma \in S_n$ , we define  $v_{T\sigma} = 0$  if  $v_{T\sigma}$  is not a standard tableau.

### 4. The relation $s_1 e_1 s_1 e_1 = e_1 s_1 e_1 s_1$

The next relation we establish is  $v_T s_1 e_1 s_1 e_1 = v_T e_1 s_1 e_1 s_1$ . The box containing 1 must be at the top left corner of one of the partitions  $\lambda^{(k)}$ . The box containing 2 can only be underneath the box containing 1, to the right of the box containing 1, or at the top left corner of the other partition  $\lambda^{(l)}$ . In each of these case we see that  $v_T s_1 = \pm v_T$ . Thus, for simplicity we write  $v_T s_1 = (-1)^a v_T$  for some integer  $a$ . Also, depending on whether the box containing 1 is in the partition  $\lambda^{(1)}$  or  $\lambda^{(2)}$ , we have  $v_T e_1 = \pm v_T$ . Again for simplicity, we write  $v_T e_1 = (-1)^b v_T$  for some integer  $b$ . Then

$$\begin{aligned}
v_T s_1 e_1 s_1 e_1 &= (-1)^a v_T e_1 s_1 e_1 = (-1)^a (-1)^b v_T s_1 e_1 = (-1)^b v_T e_1 = v_T \\
&= (-1)^a v_T s_1 = (-1)^b (-1)^a v_T e_1 s_1 = (-1)^b v_T s_1 e_1 s_1 = v_T e_1 s_1 e_1 s_1,
\end{aligned}$$

as needed.

### 5. The relation $s_i s_{i+1} s_i = s_{i+1} s_i s_{i+1}$

The last relation from the list in (1) we wish to establish is  $v_T s_i s_{i+1} s_i = v_T s_{i+1} s_i s_{i+1}$ , for  $1 \leq i \leq n-2$ . Here we have the following cases regarding the relative positions of boxes containing  $i$  and  $i+1$  and  $i+2$ :

- (a) the three boxes are in one column or in one row;
- (b) only two of the boxes are adjacent;
- (c) none of the boxes is adjacent to another.

In case (a), note that  $v_T s_i$  and  $v_T s_{i+1}$  are not standard. Hence,  $C_T(i) = C_T(i+1) = \pm 1$ , depending on whether the boxes are lined up in a column or in a row. Thus, for simplicity, we write  $C_T(i) = (-1)^a$  for some integer  $a$ . Then,

$$\begin{aligned}
v_T s_i s_{i+1} s_i &= (-1)^a v_T s_{i+1} s_i = (-1)^{2a} v_T s_i = (-1)^{3a} = (-1)^{2a} v_T s_{i+1} = (-1)^a v_T s_i s_{i+1} \\
&= v_T s_{i+1} s_i s_{i+1}.
\end{aligned}$$

In case (b), consider first the case when the boxes contained  $i$  and  $i+1$  are adjacent, and again write  $C_T(i) = (-1)^a$  for some integer  $a$ . Then  $U = \mathbb{C}\text{-span}\{v_T, v_{Ts_{i+1}}, v_{Ts_{i+1}s_i}\} \subseteq V^\lambda$  contains all the different scenarios satisfying the condition in case (b). Furthermore,  $U$  is invariant under  $H = \langle s_i, s_{i+1} \rangle$ , so it suffices to check the actions of  $H$  on  $U$ . Direct computation shows that we have the representation  $\rho : H \rightarrow \text{GL}_3(\mathbb{C})$  given by

$$s_i \mapsto \begin{pmatrix} (-1)^a & 0 & 0 \\ 0 & C_{Ts_{i+1}}(i) & 1 + C_{Ts_{i+1}}(i) \\ 0 & 1 - C_{Ts_{i+1}}(i) & -C_{Ts_{i+1}}(i) \end{pmatrix} \quad s_{i+1} \mapsto \begin{pmatrix} C_T(i) & 1 + C_T(i) & 0 \\ 1 - C_T(i) & -C_T(i) & 0 \\ 0 & 0 & (-1)^a \end{pmatrix}.$$

It follows that  $v_{Ts_i s_{i+1} s_i} = v_{Ts_{i+1} s_i s_{i+1}}$ .

We prove case (c) similarly. Let  $U = \mathbb{C}\text{-span}\{v_T, v_{Ts_i}, v_{Ts_{i+1}}, v_{Ts_i s_{i+1}}, v_{Ts_i s_{i+1} s_i}, v_{Ts_i s_{i+1} s_i s_i}\}$  and  $H$  as above. Then  $U$  is invariant under  $H$ . The representation we seek is  $\rho : H \rightarrow \text{GL}_6(\mathbb{C})$  given by

$$s_i \mapsto \begin{pmatrix} C_T(i) & 1 + C_T(i) & 0 & 0 & 0 & 0 \\ 1 - C_T(i) & -C_T(i) & 0 & 0 & 0 & 0 \\ 0 & 0 & C_{Ts_{i+1}}(i) & 0 & 1 + C_{Ts_{i+1}}(i) & 0 \\ 0 & 0 & 0 & C_{Ts_i s_{i+1}}(i) & 0 & 1 + C_{Ts_i s_{i+1}}(i) \\ 0 & 0 & 1 - C_{Ts_{i+1}}(i) & 0 & -C_{Ts_{i+1}}(i) & 0 \\ 0 & 0 & 0 & 1 - C_{Ts_i s_{i+1}}(i) & 0 & -C_{Ts_i s_{i+1}}(i) \end{pmatrix}$$

$$s_{i+1} \mapsto \begin{pmatrix} C_{Ts_i s_{i+1}}(i) & 0 & 1 + C_{Ts_i s_{i+1}}(i) & 0 & 0 & 0 \\ 0 & C_{Ts_{i+1}}(i) & 0 & 1 + C_{Ts_{i+1}}(i) & 0 & 0 \\ 1 - C_{Ts_{i+1}}(i) & 0 & -C_{Ts_{i+1}}(i) & 0 & 0 & 0 \\ 0 & 1 - C_{Ts_{i+1}}(i) & 0 & -C_{Ts_{i+1}}(i) & 0 & 0 \\ 0 & 0 & 0 & 0 & C_T(i) & 1 + C_T(i) \\ 0 & 0 & 0 & 0 & 1 - C_T(i) & -C_T(i) \end{pmatrix}.$$

And again it follows that  $v_{Ts_i s_{i+1} s_i} = v_{Ts_{i+1} s_i s_{i+1}}$  to complete the proof of the lemma.  $\square$

The next lemma we prove will assist us in proving item (ii) described in the outline above. We first define a generalization to the Murphy elements, which we will denote by  $\mu_1, \dots, \mu_n$ . These elements are members of the group algebra  $\mathbb{C}G(2, 1, n)$ . We will show that their actions on a standard tableau  $T$  of shape  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  completely determines the filling of  $T$ .

**Definition 7.** For  $k : 1, \dots, n$ , define the *generalized Murphy element*  $\mu_k \in \mathbb{C}G(2, 1, n)$  to be  $\mu_k = ke_k + \frac{1}{2} \sum_{j=1}^{k-1} (1 + e_j e_k)(j, k)$ , where  $(j, k) \in S_n$  is a transposition, and the sum is taken to be zero if  $k = 1$ .

**Lemma 8.** Let  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  be an order pair of partitions with  $n$  total boxes, and consider the  $G(2, 1, n)$ -module  $V^\lambda$  as defined above. Then for any  $v_T \in V^\lambda$ , where  $T$  is a standard tableau of

shape  $\lambda$ , and for any generalized Murphy element  $\mu_k$ , we have

$$v_T \mu_k = D_T(k) v_T, \quad (2)$$

where we define  $D_T(k)$  as follows:

$$D_T(k) = \begin{cases} (C(T(k)) + k) & \text{if the box containing } k \text{ is in } \lambda^{(1)} \\ (C(T(k)) - k) & \text{otherwise} \end{cases}.$$

Furthermore, the sequence  $D_T(1), \dots, D_T(n)$  uniquely determines the tableau  $T$ .

*Proof.* Note that for  $i : 1, \dots, k-2$  we have  $s_{k-1}(1 + e_i e_{k-1})(i, k-1) s_{k-1} = (1 + e_i e_k)(i, k)$  and  $s_{k-1}(k-1) e_{k-1} s_{k-1} = (k-1) e_k$ . Thus,

$$\mu_k = s_{k-1} \mu_{k-1} s_{k-1} + \frac{1}{2}(1 + e_{k-1} e_k) s_{k-1} + e_k.$$

We prove the first part of the lemma by induction. By definition,  $\mu_1 = e_1$  so  $v_T \mu_1 = v_T$  if the box containing 1 is in  $\lambda^{(1)}$  and  $v_T \mu_1 = -v_T$  otherwise. Since we must have  $C(T(1)) = 0$ , the base case is established. In the following, suppose that the lemma is proven for  $\mu_{k-1}$ .

Suppose that the boxes containing  $k$  and  $k-1$  are in different partitions  $\lambda^{(1)}$  and  $\lambda^{(2)}$ , not necessarily respectively. Then  $v_T \left[ \frac{1}{2}(1 + e_{k-1} e_k) \right] = \frac{1}{2}(v_T - v_T) = 0$ , and also, by Remark 5, we know that  $v_{T s_{k-1}}$  is standard. If the box containing  $k$  is in the partition  $\lambda^{(1)}$ , then

$$\begin{aligned} v_T \mu_k &= v_T \left[ s_{k-1} \mu_{k-1} s_{k-1} + \frac{1}{2}(1 + e_{k-1} e_k) s_{k-1} + e_k \right] = v_T (s_{k-1} \mu_{k-1} s_{k-1} + e_k) \\ &= v_{T s_{k-1}} \mu_{k-1} s_{k-1} + v_T = (C(T(k)) + k - 1) v_{T s_{k-1}} s_{k-1} + v_T = (C(T(k)) + k) v_T. \end{aligned}$$

Otherwise, if  $k$  is in the partition  $\lambda^{(2)}$ , then

$$\begin{aligned} v_T \mu_k &= v_T \left[ s_{k-1} \mu_{k-1} s_{k-1} + \frac{1}{2}(1 + e_{k-1} e_k) s_{k-1} + e_k \right] = v_T (s_{k-1} \mu_{k-1} s_{k-1} + e_k) \\ &= v_{T s_{k-1}} \mu_{k-1} s_{k-1} - v_T = (C(T(k)) - k + 1) v_{T s_{k-1}} s_{k-1} - v_T = (C(T(k)) - k) v_T, \end{aligned}$$

as needed.

Now suppose that boxes containing  $k$  and  $k-1$  are in the same partition  $\lambda^{(l)}$ . Then we have  $v_T \left[ \frac{1}{2}(1 + e_{k-1} e_k) \right] = \frac{1}{2}(v_T + v_T) = v_T$ . Suppose first that  $l = 1$ . Then,

$$\begin{aligned} v_T \mu_k &= v_T (s_{k-1} \mu_{k-1} s_{k-1} + s_{k-1} + e_k) = v_T [(s_{k-1} \mu_{k-1} + 1) s_{k-1} + e_k] \\ &= \left( \frac{1}{C(T(k)) - C(T(k-1))} v_T \mu_{k-1} + \left( 1 + \frac{1}{C(T(k)) - C(T(k-1))} \right) v_{T s_{k-1}} \mu_{k-1} + v_T \right) s_{k-1} + v_T \\ &= \left( \frac{C(T(k-1)) + k - 1}{C(T(k)) - C(T(k-1))} v_T + (C(T(k)) + k - 1) \left( 1 + \frac{1}{C(T(k)) - C(T(k-1))} \right) v_{T s_{k-1}} + v_T \right) s_{k-1} + v_T \\ &= \left( \frac{C(T(k)) + k - 1}{C(T(k)) - C(T(k-1))} v_T + (C(T(k)) + k - 1) \left( 1 + \frac{1}{C(T(k)) - C(T(k-1))} \right) v_{T s_{k-1}} \right) s_{k-1} + v_T \\ &= (C(T(k)) + k - 1) v_{T s_{k-1}} s_{k-1} + v_T = (C(T(k)) + k) v_T. \end{aligned}$$

If, on the other hand,  $l = 2$ , then

$$\begin{aligned}
v_T \mu_k &= v_T (s_{k-1} \mu_{k-1} s_{k-1} + s_{k-1} + e_k) = v_T [(s_{k-1} \mu_{k-1} + 1) s_{k-1} + e_k] \\
&= \left( \frac{1}{C(T(k)) - C(T(k-1))} v_T \mu_{k-1} + \left( 1 + \frac{1}{C(T(k)) - C(T(k-1))} \right) v_T s_{k-1} \mu_{k-1} + v_T \right) s_{k-1} - v_T \\
&= \left( \frac{C(T(k-1)) - k + 1}{C(T(k)) - C(T(k-1))} v_T + (C(T(k)) - k + 1) \left( 1 + \frac{1}{C(T(k)) - C(T(k-1))} \right) v_T s_{k-1} + v_T \right) s_{k-1} - v_T \\
&= \left( \frac{C(T(k)) - k + 1}{C(T(k)) - C(T(k-1))} v_T + (C(T(k)) - k + 1) \left( 1 + \frac{1}{C(T(k)) - C(T(k-1))} \right) v_T s_{k-1} \right) s_{k-1} - v_T \\
&= (C(T(k)) - k + 1) v_T s_{k-1} s_{k-1} - v_T = (C(T(k)) - k) v_T,
\end{aligned}$$

to complete the proof by induction of the first part of the lemma.

Consider any  $1 \leq k \leq n$ . The minimal value  $C(T(k))$  can attain is  $-k + 1$ , which is possible only if all the numbers  $1, \dots, k$  are placed in boxes of the same column in either  $\lambda^{(1)}$  or  $\lambda^{(2)}$ . Similarly, the maximal value  $C(T(k))$  can attain is  $k - 1$ , which is possible only if all the numbers  $1, \dots, k$  are placed in boxes of the same row in either  $\lambda^{(1)}$  or  $\lambda^{(2)}$ . Thus,  $D_T(k)$  is bounded between 1 and  $2k - 1$  if  $k$  is placed in a box in  $\lambda^{(1)}$  and is bounded between  $-2k + 1$  and  $-1$  otherwise. That is, knowing the value of  $D_T(k)$  determines in which partition  $k$  is placed. However, since the tableau  $T$  is standard, in any given partition  $\lambda^{(l)}$  we know that no two different possible placing of the box containing  $k$  can have the same content  $C(T(k))$ . The second part of the lemma follows.  $\square$

While the previous lemma showed that the basis elements  $v_T$  are eigenvectors of the generalized Murphy elements  $\mu_k$ , the next lemma shows that, up to scaling, they are the only such eigenvectors. The two lemmas together will be crucial in proving that two modules  $V^\lambda$  and  $V^\gamma$  are isomorphic only if  $\lambda = \gamma$ .

**Lemma 9.** *Let  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  be an order pair of partitions with  $n$  total boxes, and consider the  $G(2, 1, n)$ -module  $V^\lambda$  as defined above. If  $v \in V^\lambda$  is an eigenvector for all generalized Murphy elements  $\mu_1, \dots, \mu_n$  then  $v = cv_T$  for some standard tableau  $T$  of shape  $\lambda$  and complex number  $c$ .*

*Proof.* Let  $v = \sum_T a_T v_T$ , where we sum over the standard tableaux of shape  $\lambda$  and the coefficients are from the complex field. By hypothesis, we have  $v \mu_k = \alpha_k v$  for a complex number  $\alpha_k$ . On the other hand, from equation (2) we know that  $v \mu_k = \sum_T a_T D(T(k)) v_T$ . Thus, all but one  $a_T$  must vanish, and  $c = \alpha_k = D(T(k))$ .  $\square$

The last lemma we provide shows that standard tableaux form a sequence of tableaux differing by a single transposition. We will later construct a submodule of  $V^\lambda$  and show that it contains one of the basis elements  $v_T$ . The following lemma will assist us in showing that, in fact, all the basis elements are contained in the submodule, proving that the submodule is the entire module  $V^\lambda$ . We precede the lemma with an extension to the definition of a *column reading tableau* to the case of an ordered pair of partitions.

**Definition 10.** Let  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  be a partition of  $n$  total boxes. Then the *column reading tableau of shape  $\lambda$*  is a tableau  $C$  of shape  $\lambda$  satisfying the following conditions: (a) its  $\lambda^{(1)}$ -filling

is the same as the column reading tableau of shape  $\lambda^{(1)}$ ; and (b) its  $\lambda^{(2)}$ -filling is the same as the column reading tableau of shape  $\lambda^{(2)}$ , but with each entry  $k$  replaced by  $k + |\lambda^{(1)}|$ .

**Lemma 11.** *If  $P$  and  $Q$  are standard tableaux of shape  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$ , then there exists a sequence of standard tableaux  $P, P_{s_{i_1}}, P_{s_{i_1}, s_{i_2}}, \dots, P_{s_{i_1}, s_{i_2}, \dots, s_{i_r}} = Q$ .*

*Proof.* Consider the tableau  $P$ . Let  $i_1$  be the maximal number such that either (a) the box containing  $i$  is strictly above and strictly to the right of the box containing  $i_1 + 1$ ; or (b) the box containing  $i$  is in  $\lambda^{(2)}$ , while the box containing  $i_1 + 1$  is in  $\lambda^{(1)}$ . If no such  $i_1$  exists, then  $P = C$ , the column reading tableau of  $\lambda$ .

Otherwise, note the  $P_{s_{i_1}}$  is a standard tableau as well. Again find a maximal number with respect to  $P_{s_{i_1}}$  satisfying the conditions above and denote by  $i_2$ . Continue in this manner to generate  $i_j$ . The column reading tableau is characterized by the conditions above, and by Lemma 8 it is the only tableau satisfying these conditions. Furthermore, by choosing the maximal element to satisfy the conditions above, we guarantee that at no point will we arrive at a standard tableau we have seen already. Thus, the algorithm must terminate for some  $P_{s_{i_1}, s_{i_2}, \dots, s_{i_s}} = C$ .

Apply the same algorithm to  $Q$  to find  $Q_{s_{j_1}, s_{j_2}, \dots, s_{j_t}} = C$ . Then we have found a sequence  $P, P_{s_{i_1}}, \dots, P_{s_{i_1}, s_{i_2}, \dots, s_{i_r}} = C = Q_{s_{j_1}, s_{j_2}, \dots, s_{j_t}}, \dots, Q_{s_{j_1}}, Q$  of standard tableaux, as needed.  $\square$

We are now ready to prove the classification theorem.

*Proof (of the classification theorem, Theorem 4).* As mentioned above, Lemma 6 proves item (i) of the outline: the  $V^\lambda$  are  $G(2, 1, n)$ -modules.

Consider any two partitions  $\lambda = (\lambda^{(1)}, \lambda^{(2)})$  and  $\gamma = (\gamma^{(1)}, \gamma^{(2)})$  with  $n$  total boxes. Then we need to show that  $V^\lambda$  and  $V^\gamma$  are isomorphic as  $G(2, 1, n)$ -modules if and only if  $\gamma = \lambda$ ; this is item (ii) of the outline. The right-to-left implication is trivial. Conversely, suppose  $\theta : V^\lambda \rightarrow V^\gamma$  is an isomorphism of  $G(2, 1, n)$ -modules. Consider a standard tableau  $T$  of shape  $\lambda$ . Since  $\theta$  is a homomorphism, we have  $C(T(k))(v_T)\theta = (C(T(k))v_T)\theta = (v_T\mu_k)\theta = ((v_T)\theta)\mu_k$ . Hence, by Lemma 9 we know that  $(v_T)\theta = cv_Q$  for a standard tableau  $Q$  and some complex number  $c$ . By Lemma 8 we know that the sequence  $D(T(1)), \dots, D(T(n))$  and  $D(Q(1)), \dots, D(Q(n))$  uniquely determine  $T$  and  $Q$ , which by the isomorphism  $\theta$  we conclude to be the same. This completes the proof of (ii).

We have established that the  $V^\lambda$  defined in the statement of the theorem are non-isomorphic  $G(2, 1, n)$ -modules. Next we prove item (iii): that each such  $V^\lambda$  is irreducible. Fix some nonzero  $v_0 = \sum_T a_T v_T \in V^\lambda$ , and let  $M = \{v_0 x \mid x \in \mathbb{C}G(2, 1, n)\}$ , a submodule of  $V^\lambda$ . We show that  $M = V$ , which in turn implies that the only nonzero submodule of  $V^\lambda$  is itself, or equivalently, that  $V^\lambda$  is irreducible.

For each standard tableau  $T$ , consider the element

$$\pi_T = \prod_{\substack{-2n \leq j \leq 2n \\ j \neq D_T(1)}} \frac{\mu_1 - j}{D_T(1) - j} \prod_{\substack{-2n \leq j \leq 2n \\ j \neq D_T(2)}} \frac{\mu_2 - j}{D_T(2) - j} \cdots \prod_{\substack{-2n \leq j \leq 2n \\ j \neq D_T(n)}} \frac{\mu_n - j}{D_T(n) - j}.$$



If we let  $\pi_T$  act on a basis element  $v_Q$  for  $Q \neq T$ , then the product vanishes due to  $j$  assuming a value  $D(Q(k))$  for a  $k$  on which  $Q$  and  $T$  differ. On the other hand, if we let  $\pi_T$  act on  $v_T$ , then the ratios in the product are all 1, so  $\pi_T$  fixes  $v_T$ . That is, we have proved

$$v_Q \pi_T = \begin{cases} 0 & \text{if } Q \neq T \\ v_Q & \text{otherwise} \end{cases}. \quad (3)$$

Consider again  $v_0 = \sum_T a_T v_T \in V^\lambda$ . From equation (3), for each nonzero  $a_T$  we can write  $v_T = v_0 \left( \frac{\pi_T}{a_T} \right) \in M$ . By construction, there exists such a nonzero  $a_Q$ , as otherwise  $v_0 = 0$ . In particular, we have shown that at least one basis element  $v_Q$  is a member of  $M$ .

Suppose  $Qs_i$  is a standard tableau for some  $s_i$ . If the boxes containing  $i$  and  $i+1$  are in different partitions  $\lambda^{(1)}$  and  $\lambda^{(2)}$ , then  $v_{Qs_i} = v_{Qs_i} \in M$ . Otherwise, the boxes containing  $i$  and  $i+1$  are in the same partitions  $\lambda^{(l)}$ . Then  $v_{Qs_i} = \frac{1}{C(Q(i+1)) - C(Q(i))} v_Q + \left( 1 + \frac{1}{C(Q(i+1)) - C(Q(i))} \right) v_{Qs_i}$ . Note that it is impossible that  $C(Q(i+1)) - C(Q(i)) = -1$ , as this would imply that the two boxes are adjacent, contradicting the hypothesis that  $Qs_i$  is a standard tableau. It follows that  $v_{Qs_i} \in M$ . By Lemma 11 and induction on the number of transpositions, we have that  $v_T \in M$  for all standard tableaux  $T$ . But then  $M = V^\lambda$ , so  $V^\lambda$  is irreducible to prove (iii).

To prove the last item (iv), we use an indirect counting argument. Let  $G$  be any finite group. In [2], Theorem 15.3, it is proven that the number of irreducible  $G$ -modules is equal to the number of conjugacy classes of  $G$ . In [1], it is shown that the conjugacy classes of  $G(2, 1, n)$  are indexed by ordered pairs of partitions  $(\lambda^{(1)}, \lambda^{(2)})$  of  $n$  total boxes. Thus, there are no other irreducible  $G(2, 1, n)$ -modules than the ones defined in the theorem. This concludes the proof of the theorem.  $\square$

Using Theorem 4, it is possible to derive the character table of different groups  $G(2, 1, n)$  for an  $n > 2$ . We analyze the group of  $G(2, 1, 3)$ . We choose the following representatives for the conjugacy classes of  $G(2, 1, 3)$ :

$$\begin{array}{lllll} g_1 = 1 & g_2 = e_1 e_2 e_3 & g_3 = e_1 e_2 & g_4 = e_1 & g_5 = s_1 \\ g_6 = e_1 s_1 & g_7 = e_3 s_1 & g_8 = e_1 e_3 s_1 & g_9 = s_1 s_2 & g_{10} = e_1 s_1 s_2. \end{array}$$

The corresponding sizes of the centralizers  $|C_G(g_r)|$  are as follows:

$$\begin{array}{lllll} |C_G(g_1)| = 48 & |C_G(g_2)| = 48 & |C_G(g_3)| = 16 & |C_G(g_4)| = 16 & |C_G(g_5)| = 8 \\ |C_G(g_6)| = 8 & |C_G(g_7)| = 8 & |C_G(g_8)| = 8 & |C_G(g_9)| = 6 & |C_G(g_{10})| = 6. \end{array}$$

The character table of  $G(2, 1, 3)$  is the following:

	$g_1$	$g_2$	$g_3$	$g_4$	$g_5$	$g_6$	$g_7$	$g_8$	$g_9$	$g_{10}$
$ C_G(g_r) $	48	48	16	16	8	8	8	8	6	6
$\chi_1$	1	1	1	1	1	1	1	1	1	1
$\chi_2$	1	1	1	1	-1	-1	-1	-1	1	1
$\chi_3$	1	-1	1	-1	1	-1	-1	1	1	-1
$\chi_4$	1	-1	1	-1	-1	1	1	-1	1	-1
$\chi_5$	2	2	2	2	0	0	0	0	-1	-1
$\chi_6$	2	-2	2	-2	0	0	0	0	-1	1
$\chi_7$	3	-3	-1	1	1	1	-1	-1	0	0
$\chi_8$	3	-3	-1	1	-1	-1	1	1	0	0
$\chi_9$	3	3	-1	-1	1	-1	1	-1	0	0
$\chi_{10}$	3	3	-1	-1	-1	1	-1	1	0	0

We finish this discussion with a conjecture of the classification theorem of  $G(p, 1, n) \cong C_p \wr S_n$ , where  $n, p \geq 2$ . The term a *standard tableau of shape*  $\lambda$  and the notation  $C(T(i))$  appearing in this conjecture are the natural generalizations of the terms defined in Definitions 2 and 3.

**Conjecture 12.** *Let  $G(p, 1, n)$  be the monomial matrix group of  $n \times n$  matrices with exactly one nonzero entry in every column and row, and the nonzero entries are in  $C_p$ , the cyclic group of order  $p$ .*

1. *The irreducible  $G(2, 1, n)$ -modules  $V^\lambda$  are indexed by  $p$ -tuples  $\lambda = (\lambda^{(1)}, \dots, \lambda^{(p)})$  of partitions with  $n$  total boxes.*
2. *Their dimensions are  $\dim(V^\lambda) = n_\lambda$ , where  $n_\lambda$  is the number of standard tableaux of shape  $\lambda$ .*
3. *If  $V^\lambda = \mathbb{C}$ -span $\{v_T \mid T \text{ is a standard tableau of shape } \lambda\}$ , then*

$$v_T e_i = e^{\frac{2\pi i r}{p}} v_T,$$

$$v_T s_i = C_T(i) v_T + (1 + C_T(i)) v_{T s_i},$$

where (i) the box containing  $i$  is in the partition  $\lambda^{(r)}$  for some  $1 \leq r \leq p$ ; (ii)  $v_{T s_i} = 0$  if  $v_{T s_i}$  is not a standard tableau; and (iii)  $C_T(i) = \frac{1}{C(T(i+1)) - C(T(i))}$  if the boxes containing  $i$  and  $i + 1$  are in the same partition  $\lambda^{(k)}$  and zero otherwise.

## References

- [1] Ben Galin, *Math 156: Homework 4*, Spring 2007, Stanford University.
- [2] Gordon James and Martin Liebeck, *Representations and characters of groups*, 2nd ed., Cambridge University Press, Cambridge, UK, 2006.